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2000 J. Phys. A: Math. Gen. 33 795

(http://iopscience.iop.org/0305-4470/33/4/312)

J. Phys. A: Math. Gen. **33** (2000) 795–804. Printed in the UK PII: S0305-4470(00)07220-6

Weyl–Underhill–Emmrich quantization and the Stratonovich–Weyl quantizer

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Received 26 August 1999

Abstract. Weyl–Underhill–Emmrich (WUE) quantization and its generalization are considered. It is shown that an axiomatic definition of the Stratonovich–Weyl quantizer leads to severe difficulties. Quantization on the cylinder within the WUE formalism is discussed.

1. Introduction

Deformation quantization introduced in 1978 by Bayen *et al* [1] now seems to be one of the most interesting parts of mathematical physics, especially after the works of Fedosov [2, 3] and Kontsevich [4] were published. From the physical point of view the important question is whether the mathematical formalism of deformation quantization describes physical reality. One way to deal with this problem is to look for the 'natural' generalization of the Weyl–Wigner–Moyal formalism to a Riemannian configuration space and then compare this with the general theory of deformation quantization. Perhaps the most natural generalization of the Weyl quantization rule [1, 5–9] was given by Underhill [10] and Emmrich [11]. In section 2 we deal with the Weyl–Underhill–Emmrich (WUE) approach and some of its generalizations. Then we consider how this approach leads to the definition of a Stratonovich–Weyl (SW) quantizer. This quantizer is used by some authors [8, 12–15] as the fundamental object defining the deformation quantization. We argue that the axiomatic approach to the SW quantizer seems to lead to severe difficulties (see also [16]). In section 3 some aspects of deformation quantization on the cylinder within the WUE formalism are considered. It is shown how in this formalism one can define the discrete SW quantizer given by Mukunda [17] and then also obtained in [16, 18, 19].

2. WUE quantization and its generalization

First assume that the configuration space of a dynamical system is the Euclidean manifold $\mathbb{R}^n$. Then the phase space is $\mathbb{R}^{2n}$ with the natural symplectic form

$$\omega = \mathrm{d}p_\alpha \wedge \mathrm{d}x^\alpha \qquad \alpha = 1, \ldots, n \tag{2.1}$$

where $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n$ are the Cartesian coordinates on $\mathbb{R}^n$ and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ denote the respective momenta. According to the Weyl quantization rule [1, 5–9] if $f = f(\boldsymbol{p}, \boldsymbol{x})$ is a function on $\mathbb{R}^{2n}$ then the corresponding operator $\widehat{f_W}$ in the space of quantum states $\mathcal{H}$ is given by

$$\widehat{f_W} := \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}}{(2\pi\hbar)^n} f(\boldsymbol{p}, \boldsymbol{x})\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x}) \tag{2.2}$$

where $\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x} := \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_n\, \mathrm{d}\boldsymbol{x}^1 \ldots \mathrm{d}\boldsymbol{x}^n$ and the operator-valued function $\widehat{\Omega} = \widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})$ is defined by

$$\widehat{\Omega} = \widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x}) := 2^n \int_{\mathbb{R}^n} \mathrm{d}\xi \, \exp\left(-\frac{2\mathrm{i}\boldsymbol{p}\xi}{\hbar}\right)|\boldsymbol{x} - \xi\rangle\langle\boldsymbol{x} + \xi| \qquad \boldsymbol{p}\xi := \boldsymbol{p}_\alpha \xi^\alpha \tag{2.3}$$

where $\widehat{\Omega}$ is called the *Stratonovich–Weyl quantizer* [8, 9, 12–15]. One can quickly show that

$$\left\{\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})\right\}^\dagger = \widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x}) \tag{2.4}$$

$$\mathrm{Tr}\left\{\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})\right\} = 1 \tag{2.5}$$

and

$$\mathrm{Tr}\left\{\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})\widehat{\Omega}(\boldsymbol{p}', \boldsymbol{x}')\right\} = (2\pi\hbar)^n \delta(\boldsymbol{p} - \boldsymbol{p}')\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{2.6}$$

The last formula, (2.6), enables us to find the function $f = f(\boldsymbol{p}, \boldsymbol{x})$ from its Weyl image $\widehat{f_W}$. Indeed, equations (2.2) and (2.6) give

$$f = f(\boldsymbol{p}, \boldsymbol{x}) = \mathrm{Tr}\left\{\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})\widehat{f_W}\right\}. \tag{2.7}$$

Given any kets $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$ one obtains from (2.2) and (2.3)

$$\begin{aligned}
\langle\varphi|\widehat{f_W}|\psi\rangle &= \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}}{(2\pi\hbar)^n} f(\boldsymbol{p}, \boldsymbol{x})\langle\varphi|\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})|\psi\rangle \\
\langle\varphi|\widehat{\Omega}(\boldsymbol{p}, \boldsymbol{x})|\psi\rangle &= 2^n \int_{\mathbb{R}^n} \mathrm{d}\xi \, \exp\left(-\frac{2\mathrm{i}\boldsymbol{p}\xi}{\hbar}\right)\overline{\varphi(\boldsymbol{x} - \xi)}\, \psi(\boldsymbol{x} + \xi)
\end{aligned} \tag{2.8}$$

where $\varphi(\boldsymbol{x}) = \langle\boldsymbol{x}|\varphi\rangle$ and $\psi(\boldsymbol{x}) = \langle\boldsymbol{x}|\psi\rangle$ denote the Schrödinger representation of $|\varphi\rangle$, and $|\psi\rangle$, respectively, and the overbar denotes the complex conjugation. Finally,

$$\langle\varphi|\widehat{f_W}|\psi\rangle = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^{2n}\times\mathbb{R}^n} \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\xi \; f(\boldsymbol{p}, \boldsymbol{x}) \exp\left(-\frac{2\mathrm{i}\boldsymbol{p}\xi}{\hbar}\right)\overline{\varphi(\boldsymbol{x} - \xi)}\psi(\boldsymbol{x} + \xi). \tag{2.9}$$

In particular, let f be a monomial in momenta

$$f = X^{\alpha_1\ldots\alpha_m}(\boldsymbol{x})\boldsymbol{p}_{\alpha_1} \ldots \boldsymbol{p}_{\alpha_m} \tag{2.10}$$

where $X^{\alpha_1\ldots\alpha_m}(\boldsymbol{x})$ is a totally symmetric tensor field on the configuration space $\mathbb{R}^n$. Substituting (2.10) into (2.9), integrating with respect to $\mathrm{d}\boldsymbol{p}$ and then by parts with respect to $\mathrm{d}\xi$ we obtain

$$\begin{aligned}
\langle\varphi|\widehat{f_W}|\psi\rangle &= \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\xi\, X^{\alpha_1\ldots\alpha_m}(\boldsymbol{x})\left(-\frac{\hbar}{2\mathrm{i}}\right)^m \\
&\quad \times \overline{\varphi(\boldsymbol{x} - \xi)}\psi(\boldsymbol{x} + \xi)\frac{\partial^m}{\partial\xi^{\alpha_1} \ldots \partial\xi^{\alpha_m}}\left\{(2\pi)^n \delta\left(\frac{2\xi}{\hbar}\right)\right\} \\
&= \left(\frac{\hbar}{2\mathrm{i}}\right)^m \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{x}\, X^{\alpha_1\ldots\alpha_m}(\boldsymbol{x})\frac{\partial^m}{\partial\xi^{\alpha_1} \ldots \partial\xi^{\alpha_m}}\left\{\overline{\varphi(\boldsymbol{x} - \xi)}\psi(\boldsymbol{x} + \xi)\right\}_{\xi=0}.
\end{aligned} \tag{2.11}$$

Finally, the integration by parts brings (2.11) to the form

$$\langle\varphi|\widehat{f}_W|\psi\rangle = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{x}\,\overline{\varphi(\boldsymbol{x})}\Bigg\{\left(\frac{\hbar}{\mathrm{i}}\right)^m \sum_{k=0}^{m} \frac{1}{2^k}\binom{m}{k} \times\big(\partial_{\alpha_1}\ldots\partial_{\alpha_k}X^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_m}(\boldsymbol{x})\big)\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_m}\Bigg\}\psi(\boldsymbol{x}). \tag{2.12}$$

Consequently, the Weyl image of the monomial (2.10) reads

$$\widehat{f}_W = \left(\frac{\hbar}{\mathrm{i}}\right)^m \sum_{k=0}^{m} \frac{1}{2^k}\binom{m}{k}\big(\partial_{\alpha_1}\ldots\partial_{\alpha_k}X^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_m}(\boldsymbol{x})\big)\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_m}. \tag{2.13}$$

By the linear extension of (2.13) one obtains the Weyl image for an arbitrary polynomial in momenta. As has been shown in [9, 20, 21] every operator ordering satisfying some natural axioms can be obtained with the use of an operator of the form

$$A = A\left(-\hbar\frac{\partial^2}{\partial\boldsymbol{p}_\alpha\partial\boldsymbol{x}^\alpha}\right) = 1 + \sum_{k=1}^{\infty} A_k\cdot\left(-\hbar\frac{\partial^2}{\partial\boldsymbol{p}_\alpha\partial\boldsymbol{x}^\alpha}\right)^k \qquad A_k\in\mathbb{C}. \tag{2.14}$$

Given operator A one defines

$$\begin{aligned}\widehat{f}^{(A)} &:= \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}}{(2\pi\hbar)^n}(Af(\boldsymbol{p},\boldsymbol{x}))\widehat{\Omega}(\boldsymbol{p},\boldsymbol{x})\\ &= \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}}{(2\pi\hbar)^n}f(\boldsymbol{p},\boldsymbol{x})\widehat{\Omega}^{(A)}(\boldsymbol{p},\boldsymbol{x})\end{aligned} \tag{2.15}$$

where

$$\widehat{\Omega}^{(A)}(\boldsymbol{p},\boldsymbol{x}) := A\widehat{\Omega}(\boldsymbol{p},\boldsymbol{x}) \tag{2.16}$$

is called the *generalized Stratonovich–Weyl quantizer* [9]. We have

$$\mathrm{Tr}\big\{\widehat{\Omega}^{(A)}(\boldsymbol{p},\boldsymbol{x})\big\} = 1 \tag{2.17}$$

$$\mathrm{Tr}\big\{\widehat{\Omega}^{(A)}(\boldsymbol{p},\boldsymbol{x})\widehat{\Omega}^{(A)}(\boldsymbol{p}',\boldsymbol{x}')\big\} = (2\pi\hbar)^n A^2\left(-\hbar\frac{\partial^2}{\partial\boldsymbol{p}_\alpha\partial\boldsymbol{x}^\alpha}\right)\delta(\boldsymbol{p}-\boldsymbol{p}')\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.18}$$

Hence one obtains the generalization of the formula (2.7)

$$f = f(\boldsymbol{p},\boldsymbol{x}) = A^{-2}\left(-\hbar\frac{\partial^2}{\partial\boldsymbol{p}_\alpha\partial\boldsymbol{x}^\alpha}\right)\mathrm{Tr}\big\{\widehat{\Omega}^{(A)}(\boldsymbol{p},\boldsymbol{x})\widehat{f}^{(A)}\big\}. \tag{2.19}$$

For the Weyl ordering we have

$$A = 1 \tag{2.20}$$

and for the so-called *standard ordering*

$$A = \exp\left\{\frac{\mathrm{i}\hbar}{2}\frac{\partial^2}{\partial\boldsymbol{p}_\alpha\partial\boldsymbol{x}^\alpha}\right\}. \tag{2.21}$$

In what follows we denote by $\widehat{f}_W$ and $\widehat{f}_S$ the Weyl and the standard ordering, respectively. One can easily show that if f is the monomial (2.10) then

$$\widehat{f}_S = \left(\frac{\hbar}{\mathrm{i}}\right)^m X^{\alpha_1\ldots\alpha_m}(\boldsymbol{x})\partial_{\alpha_1}\ldots\partial_{\alpha_m}. \tag{2.22}$$

It is evident that $\widehat{f}^{(A)}$ is Hermitian for every real monomial of the form (2.10) if and only if

$$\overline{A} = A. \tag{2.23}$$

Our intent is to generalize the above considerations for the case when the configuration space is an n-dimensional Riemannian manifold (M, g), where $g \in \mathrm{Symm}(T^*M \otimes T^*M)$ is the metric on M. The phase space is the cotangent bundle T^*M over M endowed with the natural symplectic form

$$\omega = \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha \qquad \alpha = 1, \ldots, n \tag{2.24}$$

where $q^1, \ldots, q^n$ are coordinates in M and $p_1, \ldots, p_n, q^1, \ldots, q^n$ are the induced coordinates (the proper Darboux coordinates) in T^*M. Let $f = f(p, q)$ be a function on T^*M. The question is to find a natural generalization of the Weyl quantization rule for $\mathbb{R}^{2n}$ to the case of T^*M. It seems that the best answer to this question has been given by Underhill [10] and then by Emmrich [11]. We follow them changing only the measure used in the integration over TM. (Concerning the Underhill–Emmrich approach also see the distinguished papers by Bordemann *et al* [22, 23] and Pflaum [24, 25]). A first glance at the formulae (2.2), (2.3) and (2.9) shows that the main problem lies in a definition of the term $\exp(-2\mathrm{i}p\xi/\hbar)$ when M is no longer the Euclidean space $\mathbb{R}^n$. In the Underhill–Emmrich approach it is done by the use of normal coordinates. Let q be any point of M and $T_q(M)$ and $T_q^*(M)$ be the tangent and cotangent space of M at q, respectively. For any $\xi = \xi^\alpha(\partial/\partial q^\alpha)_q \in T_q(M)$ and $p = p_\alpha(\mathrm{d}q^\alpha)_q \in T_q^*(M)$ we write as before $p\xi := p_\alpha\xi^\alpha$. For every $q \in M$ we choose a normal neighbourhood $V_q' \subset T_q(M)$, an open ball $K_q \subset V_q'$ and some smaller neighbourhood of q, $V_q \subset K_q$. Then one defines a cut-off function $\chi = \chi(q, \xi) \in C^\infty(TM)$ such that for every $q \in M$

$$\chi(q, \xi) = \begin{cases} 1 & \text{for} \quad \xi \in V_q \\ 0 & \text{for} \quad \xi \notin K_q. \end{cases} \tag{2.25}$$

Let $\exp_q : V_q' \longrightarrow U_q \subset M$ be the exponential map of V_q' onto U_q. For any functions φ and ψ on M and for every point $q \in M$ we define the functions Φ_q^- and Ψ_q^+ on $T_q(M)$ by

$$\begin{aligned} \Phi_q^-(\xi) &= \begin{cases} \chi(q, -\xi)\varphi(\exp_q(-\xi)) & \text{for} \quad \xi \in K_q \\ 0 & \text{for} \quad \xi \notin K_q \end{cases} \\ \Psi_q^+(\xi) &= \begin{cases} \chi(q, \xi)\psi(\exp_q \xi) & \text{for} \quad \xi \in K_q \\ 0 & \text{for} \quad \xi \notin K_q. \end{cases} \end{aligned} \tag{2.26}$$

Let $f = f(p, q)$ be a function on T^*M. By analogy with (2.9) one assigns to f the following operator $\widehat{f}_W$:

$$\langle\varphi|\widehat{f}_W|\psi\rangle := \frac{1}{(\pi\hbar)^n}\int_{T^*M} \mathrm{d}p\,\mathrm{d}q\; f(p, q)\int_{T_q(M)} \sqrt{g(\xi)}\,\mathrm{d}\xi\,\exp\left(-\frac{2\mathrm{i}p\xi}{\hbar}\right)\overline{\Phi_q^-(\xi)}\Psi_q^+(\xi). \tag{2.27}$$

Then we also have

$$\begin{aligned} \langle\varphi|\widehat{f}_W|\psi\rangle &= \int_{T^*M} \frac{\mathrm{d}p\,\mathrm{d}q}{(2\pi\hbar)^n} f(p, q)\langle\varphi|\widehat{\Omega}(p, q)|\psi\rangle \\ \langle\varphi|\widehat{\Omega}(p, q)|\psi\rangle &= 2^n\int_{T_q(M)} \sqrt{g(\xi)}\,\mathrm{d}\xi\,\exp\left(-\frac{2\mathrm{i}p\xi}{\hbar}\right)\overline{\Phi_q^-(\xi)}\Psi_q^+(\xi) \end{aligned} \tag{2.28}$$

where $g(\xi)$ stands for the determinant of the metric on M in the normal coordinates. Note that Underhill [10] assumes the measure to be $\mathrm{d}\xi$ and, consequently φ and ψ are half-densities. On the other hand, Emmrich [11] deals with the measure $\sqrt{g(\xi)}\,\mathrm{d}\xi$ and therefore φ and ψ are scalars. We assume that the wavefunctions φ and ψ are scalars but the measure on $V_q' \subset T_q(M)$ is $\sqrt{g(\xi)}\,\mathrm{d}\xi$. The operator $\widehat{\Omega}(p,q)$ defined by (2.28) now plays the role of the SW quantizer. The only problem is that both $\widehat{\Omega}$ and $\widehat{f_W}$ depend on the cut-off function $\chi(q,\xi)$. Thus one should find the 'optimal' form of χ. However, as was shown by Underhill [10], if the function f is a polynomial with respect to momenta then $\widehat{f_W}$ does not depend on χ. Indeed, let

$$f = f(p,q) = X^{\alpha_1\ldots\alpha_m}(q)\,p_{\alpha_1}\ldots p_{\alpha_m}. \tag{2.29}$$

Substituting (2.29) into (2.27), integrating with respect to $\mathrm{d}p$ and then, by parts, with respect to $\mathrm{d}\xi$ one obtains

$$\begin{aligned}\langle\varphi|\widehat{f_W}|\psi\rangle &= \frac{1}{(\pi\hbar)^n}\int_{T^*M}\mathrm{d}p\,\mathrm{d}q\,X^{\alpha_1\ldots\alpha_m}(q)\int_{T_q(M)}\sqrt{g(\xi)}\,\mathrm{d}\xi\left(-\frac{\hbar}{2\mathrm{i}}\right)^m\\ &\quad\times\left\{\frac{\partial^m}{\partial\xi^{\alpha_1}\ldots\partial\xi^{\alpha_m}}\exp\left(-\frac{2\mathrm{i}p\xi}{\hbar}\right)\right\}\overline{\Phi_q^-(\xi)}\Psi_q^+(\xi)\\ &= \left(\frac{\hbar}{2\mathrm{i}}\right)^m\int_M\sqrt{g(q)}\,\mathrm{d}q\,X^{\alpha_1\ldots\alpha_m}(q)\left\{\frac{\partial^m}{\partial\xi^{\alpha_1}\ldots\partial\xi^{\alpha_m}}\widetilde{D}(q,\xi)\right\}_{\xi=0}\end{aligned} \tag{2.30}$$

$$\widetilde{D}(q,\xi) := \frac{\sqrt{g(\xi)}}{\sqrt{g(q)}}\overline{\Phi_q^-(\xi)}\Psi_q^+(\xi).$$

However, it is an easy matter to show that (see Petrov [26])

$$\begin{aligned}\left\{\frac{\partial^k}{\partial\xi^{\alpha_1}\ldots\partial\xi^{\alpha_k}}\overline{\Phi^-(\xi)}\right\}_{\xi=0} &= (-1)^k\nabla_{(\alpha_1}\ldots\nabla_{\alpha_k)}\overline{\varphi(q)}\\ \left\{\frac{\partial^k}{\partial\xi^{\alpha_1}\ldots\partial\xi^{\alpha_k}}\Psi^+(\xi)\right\}_{\xi=0} &= \nabla_{(\alpha_1}\ldots\nabla_{\alpha_k)}\psi(q)\end{aligned} \tag{2.31}$$

where $\nabla_{\alpha_1} := \nabla_{\partial/\partial q^{\alpha_1}},\ldots$ etc, and the bracket $(\alpha_1\ldots\alpha_k)$ stands for the symmetrization. Finally, inserting (2.31) into (2.30) and integrating by parts one arrives at the following result which is a generalization of the one obtained by Bordemann *et al* [23]:

$$\begin{aligned}\langle\varphi|\widehat{f_W}|\psi\rangle &= \int_M\sqrt{g(q)}\,\mathrm{d}q\,\overline{\varphi(q)}\,\widehat{f_W}\psi(q)\\ \widehat{f_W} &= \left(\frac{\hbar}{\mathrm{i}}\right)^m\sum_{k=0}^m\binom{m}{k}\sum_{j=0}^{m-k}\binom{m-k}{j}\frac{1}{2^{k+j}}\\ &\quad\times\left(\nabla_{\alpha_1}\ldots\nabla_{\alpha_j}\widetilde{X}^{\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_{m-k}}(q)\right)\nabla_{\alpha_{j+1}}\ldots\nabla_{\alpha_{m-k}}\\ &= \sum_{k=0}^m\left(\frac{\hbar}{2\mathrm{i}}\right)^k\binom{m}{k}\left\{\left(\frac{\hbar}{\mathrm{i}}\right)^{m-k}\sum_{j=0}^{m-k}\left(\frac{1}{2}\right)^j\binom{m-k}{j}\right.\\ &\quad\left.\times\left(\nabla_{\alpha_1}\ldots\nabla_{\alpha_j}\widetilde{X}^{\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_{m-k}}(q)\right)\nabla_{\alpha_{j+1}}\ldots\nabla_{\alpha_{m-k}}\right\}\end{aligned} \tag{2.32}$$

$$\widetilde{X}^{\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_{m-k}}(q) := X^{\beta_1\ldots\beta_k\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_{m-k}}(q)\left\{\frac{\partial^k}{\partial\xi^{\beta_1}\ldots\partial\xi^{\beta_k}}\frac{\sqrt{g(\xi)}}{\sqrt{g(q)}}\right\}_{\xi=0}.$$

The term corresponding to $k=0$ is exactly the operator given in [22, 23]. (Compare also with (2.13).) Thus one concludes that if $f = f(p,q)$ is a monomial of the form (2.29) then $\widehat{f_W}$

given by (2.32) is independent of the cut-off function χ. By linearity this is also true for any polynomial with respect to momenta.

Examples. (Compare with [10, 11, 23].)

(a) Assume

$$f = X^{\alpha}(q) p_{\alpha} \tag{2.33}$$

then

$$\widehat{f}_W = \frac{\hbar}{\mathrm{i}}\big[X^{\alpha}(q)\nabla_{\alpha} + \tfrac{1}{2}\big(\nabla_{\alpha}X^{\alpha}(q)\big)\big]. \tag{2.34}$$

(b)

$$f = X^{\alpha\beta}(q) p_{\alpha} p_{\beta} \tag{2.35}$$

Here

$$\widehat{f}_W = \left(\frac{\hbar}{\mathrm{i}}\right)^2\big[X^{\alpha\beta}(q)\nabla_{\alpha}\nabla_{\beta} + \big(\nabla_{\alpha}X^{\alpha\beta}(q)\big)\nabla_{\beta} + \tfrac{1}{4}\big(\nabla_{\alpha}\nabla_{\beta}X^{\alpha\beta}(q)\big) + \tfrac{1}{12}X^{\alpha\beta}(q)R_{\alpha\beta}(q)\big] \tag{2.36}$$

where $R_{\alpha\beta}(q)$ is the Ricci tensor on M

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta} = \partial_{\gamma}\Gamma^{\gamma}_{\alpha\beta} - \partial_{\beta}\Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\delta}\Gamma^{\delta}_{\alpha\gamma}. \tag{2.37}$$

(c) Let

$$f = X^{\alpha\beta}(q)p_{\alpha}p_{\beta} + \mathrm{i}\hbar\big(\nabla_{\alpha}X^{\alpha\beta}(q)\big)p_{\beta} - \tfrac{1}{4}\hbar^2\big(\nabla_{\alpha}\nabla_{\beta}X^{\alpha\beta}(q)\big) + \tfrac{1}{12}\hbar^2 X^{\alpha\beta}(q)R_{\alpha\beta}(q). \tag{2.38}$$

Then

$$\widehat{f}_W = \left(\frac{\hbar}{\mathrm{i}}\right)^2 X^{\alpha\beta}(q)\nabla_{\alpha}\nabla_{\beta}. \tag{2.39}$$

Now we are in a position to consider an important problem. As has been mentioned, the operator $\widehat{\Omega}(p,q)$ given by (2.28) is the SW quantizer within the WUE formalism. Of course, $\widehat{\Omega}(p,q)$ depends on the cut-off function $\chi(q,\xi)$. Therefore, the question is whether there exists $\chi(q,\xi)$ such that the usual axioms of the SW quantizer [8, 13–15], i.e.

$$\{\widehat{\Omega}(p,q)\}^{\dagger} = \widehat{\Omega}(p,q) \tag{2.40}$$

$$\mathrm{Tr}\{\widehat{\Omega}(p,q)\} = 1 \tag{2.41}$$

$$\int_{T^*M}\frac{\mathrm{d}p'\,\mathrm{d}q'}{(2\pi\hbar)^n} f(p',q')\,\mathrm{Tr}\Big\{\widehat{\Omega}(p,q)\widehat{\Omega}(p',q')\Big\} = f(p,q) \iff \mathrm{Tr}\big\{\widehat{\Omega}(p,q)\widehat{f}_W\big\} = f(p,q) \tag{2.42}$$

are satisfied by the operator $\widehat{\Omega}(p,q)$ defined by (2.28) (see (2.4)–(2.7)). It is evident that the condition (2.40) holds for any $\chi(q,\xi)$. Now to check (2.41) we take a complete orthonormal system of functions $\{\varphi_j\}$ on M,

$$\begin{gathered}\int_M \sqrt{g(q)}\,\mathrm{d}q\,\overline{\varphi_{k'}(q)}\varphi_k(q) = \delta_{kk'}\\ \sum_k \overline{\varphi_k(q')}\,\varphi_k(q) = \frac{\delta(q-q')}{\sqrt{g(q)}}.\end{gathered} \tag{2.43}$$

It is an easy matter to observe that without any loss of generality one can use the exponential functions in the tangent space $T_q(M)$

$$\begin{aligned}\varphi_s(\xi) &= \frac{1}{(\sqrt{2\pi})^n \sqrt[4]{g(\xi)}} \exp(\mathrm{i}s\xi)\\ s &= (s_1, \ldots, s_n) \in \mathbb{Z} \times \cdots \times \mathbb{Z}.\end{aligned} \tag{2.44}$$

Consequently, we obtain

$$\begin{aligned}\mathrm{Tr}\{\widehat{\Omega}(p,q)\} &= \sum_{s\in\mathbb{Z}\times\cdots\times\mathbb{Z}} \langle\varphi_s|\widehat{\Omega}(p,q)|\varphi_s\rangle\\ &= 2^n \int_{T_q(M)} \sqrt{g(\xi)}\,\mathrm{d}\xi\, \frac{\chi(q,-\xi)\chi(q,\xi)}{(2\pi)^n \sqrt[4]{g(-\xi)}\sqrt[4]{g(\xi)}} (2\pi)^n \delta(2\xi) = 1.\end{aligned} \tag{2.45}$$

Remark. In (2.44) and (2.45) it is assumed that $K_q \subset [-\pi, \pi] \times \cdots \times [-\pi, \pi]$. In other cases we should change the period of the exponential functions but the final result of (2.45) holds true.

Thus (2.41) is fulfilled for every cut-off function $\chi(q, \xi)$. Consider now the condition (2.42). To this end we use example (c). Inserting the operator (2.39) into (2.42), using as before the exponential functions (2.44) and also employing some formulae from the theory of the normal coordinate systems [26] one arrives at the following result:

$$f(p,q) - \mathrm{Tr}\left\{\widehat{\Omega}(p,q)\widehat{f_W}\right\} = \tfrac{1}{3}\hbar^2 X^{\alpha\beta}(q) R_{\alpha\beta}(q) \tag{2.46}$$

where $f = f(p,q)$ is defined by (2.38). As (2.46) holds true for an arbitrary $\chi(q, \xi)$ the axiom (2.42) cannot be satisfied. One can quickly show that the analogous result to (2.46) holds true when the Emmrich measure $\sqrt{g(q)}\,\mathrm{d}\xi$ is considered. Thus we conclude that: *in general the axiom* (2.42) *is not satisfied within the WUE formalism for any choice of the cut-off function* $\chi(q, \xi)$. Therefore, from the WUE formalism point of view the axiomatic approach to the definition of the SW quantizer seems to be questionable. (See also [16, 19] and the next section of the present paper.) Finally, let us consider the problem of different operator orderings. One can quickly find that in the Euclidean case if we perform a point transformation

$$q^\alpha = q^\alpha\big(\boldsymbol{x}^\beta\big) \qquad p_\alpha = \frac{\partial \boldsymbol{x}^\beta\big(q^\gamma\big)}{\partial q^\alpha}\boldsymbol{p}_\beta \tag{2.47}$$

where $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n$ are the Cartesian coordinates and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ the corresponding momenta, then

$$-\hbar \frac{\partial^2}{\partial \boldsymbol{p}_\alpha \partial \boldsymbol{x}^\alpha} = -\hbar \left\{ \frac{\partial^2}{\partial p_\alpha \partial q^\alpha} + p_\gamma \Gamma^\gamma_{\alpha\beta}(q) \frac{\partial^2}{\partial p_\alpha \partial p_\beta} + \Gamma^\beta_{\alpha\beta}(q) \frac{\partial}{\partial p_\alpha} \right\} \tag{2.48}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols with respect to the coordinates q^α. Hence, it is natural to generalize the object A defining the operator ordering in the Euclidean case (see (2.14)) to the following one:

$$\begin{aligned}A &= A(\Delta) = 1 + \sum_{k=1}^{\infty} A_k \Delta^k \qquad A_k \in \mathbb{C}\\ \Delta &:= -\hbar \left(\frac{\partial^2}{\partial p_\alpha \partial q^\alpha} + p_\gamma \Gamma^\gamma_{\alpha\beta} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} + \Gamma^\beta_{\alpha\beta} \frac{\partial}{\partial p_\alpha} \right)\end{aligned} \tag{2.49}$$

when the configuration space is an n-dimensional Riemannian manifold $(M, \mathrm{d}s^2)$. (The operator Δ was also found by Bordemann *et al* [22, 23].) Consequently, we now have

$$\begin{aligned}\langle\varphi|\widehat{f}^{(A)}|\psi\rangle &= \int_{T^*M} \frac{\mathrm{d}p\,\mathrm{d}q}{(2\pi\hbar)^n}(Af(p,q))\langle\varphi|\widehat{\Omega}(p,q)|\psi\rangle \\ &= \int_{T^*M} \frac{\mathrm{d}p\,\mathrm{d}q}{(2\pi\hbar)^n} f(p,q)\langle\varphi|\widehat{\Omega}^{(A)}(p,q)|\psi\rangle \end{aligned} \tag{2.50}$$

where the generalized SW quantizer $\widehat{\Omega}^{(A)}(p,q)$ is defined by

$$\widehat{\Omega}^{(A)}(p,q) := A\widehat{\Omega}(p,q). \tag{2.51}$$

In particular, for the *generalized standard ordering* one puts [22]

$$A = \exp\left\{\tfrac{1}{2}\mathrm{i}\hbar\Delta\right\} \tag{2.52}$$

and for the monomial (2.29) we obtain

$$\widehat{f_S} := \widehat{f}^{(A)} = \left(\frac{\hbar}{\mathrm{i}}\right)^m \sum_{k=0}^{m} \frac{1}{2^k}\begin{pmatrix} m \\ k \end{pmatrix} \widetilde{X}^{\alpha_1\ldots\alpha_{m-k}}(q)\nabla_{\alpha_1}\ldots\nabla_{\alpha_{m-k}}. \tag{2.53}$$

The term with $k = 0$ corresponds exactly to the operator in standard ordering in the case of the Emmrich measure [22].

3. Quantization on the cylinder

Consider a simple dynamical system consisting of one particle on the circle S^1. The phase space of this system is the cylinder $\mathbb{R}\times S^1$. The deformation quantization for this case might seem to be a simple modification of the Euclidean case. However, it is not because of the non-trivial topology of S^1. In particular, one arrives at the conclusion that if the deformation quantization on the cylinder $\mathbb{R}\times S^1$ is to give 'physical' results then the classical phase space should be quantized to be $\hbar\mathbb{Z}\times S^1$ [16–19]. Here we consider some aspects of the deformation quantization on the cylinder using the Weyl–Underhill–Emmrich quantization rule. In the present case the configuration space $M = S^1$, then $T^*M = \mathbb{R}\times S^1$ and $T_qM = \mathbb{R}$. For the coordinate q we use the angle θ, $-\pi \leqslant \theta < \pi$. The complete orthonormal system of $L^2(S^1)$ is given by

$$\varphi_k = \frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}k\theta) \qquad k \in \mathbb{Z}. \tag{3.1}$$

For simplicity we assume that the cut-off function $\chi(\theta,\xi)$ is symmetric with respect to ξ

$$\chi(\theta,\xi) = \chi(\theta,-\xi) \qquad \forall\,\theta \in [-\pi,\pi[. \tag{3.2}$$

The SW quantizer $\widehat{\Omega}(p,\theta)$ defined by (2.28) now reads

$$\langle\varphi_k|\widehat{\Omega}(p,\theta)|\varphi_{k'}\rangle = \frac{1}{\pi}\exp\left\{\mathrm{i}(k'-k)\theta\right\}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\chi^2(\theta,\xi)\exp\left\{\mathrm{i}\left(k+k'-\frac{2p}{\hbar}\right)\xi\right\}. \tag{3.3}$$

One can quickly check that according to the general formula (2.45)

$$\mathrm{Tr}\{\widehat{\Omega}(p,\theta)\} = \sum_{k\in\mathbb{Z}}\langle\varphi_k|\widehat{\Omega}(p,\theta)|\varphi_k\rangle = 1 \tag{3.4}$$

for arbitrary χ. Let $f = f(p,\theta)$ be a monomial

$$f(p,\theta) = X(\theta)p^m \tag{3.5}$$

then

$$\int_{\mathbb{R}\times S^1} \frac{\mathrm{d}p'\,\mathrm{d}\theta'}{2\pi\hbar} f(p',\theta')\,\mathrm{Tr}\big\{\widehat{\Omega}(p,\theta)\widehat{\Omega}(p',\theta')\big\} = \sum_{k\in\mathbb{Z}} \langle\varphi_k|\widehat{\Omega}(p,\theta)\widehat{f}_W|\varphi_k\rangle$$
$$= \int_{-\infty}^{\infty} \mathrm{d}\xi\, \chi^2(\theta,\xi) \exp\left(-\frac{2\mathrm{i}p\xi}{\hbar}\right)\left(\frac{\hbar}{2\mathrm{i}}\right)^m \frac{\partial^m}{\partial\xi^m}(X(\theta+\xi)\delta(\xi)) = X(\theta)p^m. \tag{3.6}$$

By the linearity of the integral (2.42) with respect to f one concludes that *the axiom* (2.42) *is now satisfied for a function* $f = f(p,\theta)$ *being an arbitrary polynomial in the momentum* p. If we want the axiom (2.42) to hold for any function on the cylinder then $\mathrm{Tr}\big\{\widehat{\Omega}(p,\theta)\widehat{\Omega}(p',\theta')\big\}$ should be equal to $2\pi\hbar\delta(\theta-\theta')\delta(p-p')$. Performing simple manipulations, remembering also that $\chi(\theta,\xi) = 0$ for $\xi \neq]-\pi,\pi[$ (i.e. $K_\theta \subset]-\pi,\pi[$) one finds

$$\mathrm{Tr}\big\{\widehat{\Omega}(p,\theta)\widehat{\Omega}(p',\theta')\big\} = 2\delta(\theta-\theta')\int_{-\infty}^{\infty}\mathrm{d}\xi\,\chi^4(\theta,\xi)\exp\left\{\frac{2\mathrm{i}}{\hbar}(p'-p)\xi\right\}$$
$$+4(\delta(\theta-\theta'-\pi)+\delta(\theta-\theta'+\pi))\int_{-\infty}^{\infty}\mathrm{d}\xi\,\chi^2(\theta,\xi)\chi^2(\theta',\xi+\pi)$$
$$\times\cos\left\{\frac{2}{\hbar}(p'-p+\pi)\xi\right\}. \tag{3.7}$$

Hence, as $\chi(\theta,\xi)$ has a compact support with respect to ξ the formula (3.7) never gives $2\pi\hbar\delta(\theta-\theta')\delta(p-p')$. Consequently, *the axiom* (2.42) *cannot be satisfied for an arbitrary function on the cylinder*. (Note that this is always the case if the configuration space M is such that for some point q of M the normal coordinates at q cannot be extended to all the tangent space $T_q(M)$.) We must mention here that in the important works [27] the SW quantizer on the cylinder satisfying the axioms (2.40)–(2.42) has been found. However, this SW quantizer has a disadvantage (as does our SW quantizer (3.3)), that is, it does not fulfil the condition: $\widehat{f}_W = f(\widehat{p})$ for an arbitrary function $f = f(p)$, which could be expected for a particle on a circle. The same occurs in the interesting approach of Alcalde [28] where the notion of an SW quantizer is not used. In fact, as is known from [16] the violation of the above condition will always appear unless we consider a 'quantization' of the classical cylindrical phase space. This quantization in the WUE formalism can be obtained by some limiting process. Namely, let $\{\chi_j(\theta,\xi)\}_{j\in\mathbb{N}}$ be a series of cut-off functions such that for every $j\in\mathbb{N}$ and every $\theta\in[-\pi,\pi[$

$$0\leqslant\chi_j(\theta,\xi)\leqslant 1 \qquad \chi_j(\theta,\xi)=0 \qquad \text{for}\quad \xi\notin\left]-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi\right[\tag{3.8}$$

and

$$\lim_{j\to\infty}\int_{-\pi/2}^{\pi/2}\mathrm{d}\xi\,(\chi_j(\theta,\xi))^m f(\xi) = \int_{-\pi/2}^{\pi/2}\mathrm{d}\xi\, f(\xi) \tag{3.9}$$

for every $m\in\mathbb{N}$ and every continuous function $f = f(\xi)$ (see Vladimirov [29], section 2.2). Assuming that the momentum $p = n\hbar$, $n\in\mathbb{Z}$, using (3.8) and (3.9) one quickly finds that (3.3) leads to

$$\begin{aligned}&\lim_{j\to\infty}\langle\varphi_k|\widehat{\Omega}_j(k\hbar,\theta)|\varphi_{k'}\rangle = \frac{1}{\pi}\exp\{\mathrm{i}(k'-k)\theta\}\\ &\lim_{j\to\infty}\int_{-\pi/2}^{\pi/2}\mathrm{d}\xi\,\chi_j^2(\theta,\xi)\exp\{\mathrm{i}(k+k'-2n)\xi\} = \langle\varphi_k|\widehat{\Omega}(n,\theta)|\varphi_{k'}\rangle\end{aligned} \tag{3.10}$$

where $\widehat{\Omega}(n, \theta)$ is the *discrete Stratonovich–Weyl quantizer for the cylinder* found by Mukunda [17] and also given in [16, 18, 19]. Then from (3.7) with (3.8) and (3.9) we have

$$\lim_{j\to\infty} \mathrm{Tr}\left\{\widehat{\Omega}_j(n, \theta)\widehat{\Omega}_j(n', \theta')\right\} = 2\pi\delta_{n,n'}\delta(\theta - \theta') \tag{3.11}$$

(compare with [16]). Finally, note that the discrete SW quantizer (3.10) gives $\widehat{f}_W = f(\widehat{p})$ for every function $f = f(p)$ as is expected.

Acknowledgments

This paper was partially supported by CONACYT and CINVESTAV (México) and by KBN (Poland). One of us (MP) is grateful to the staff of Departamento de Física at CINVESTAV, (México, DF) for warm hospitality.

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